It is both more informative and more useful in calculations to consider plane-wave solutions since these correspond to states of definite momentum which is typically what we put in to scattering events and detect coming out.

Taking a plane-wave ansatz: $\psi(x) = \underbrace{A}_{\text{overall normalization constant}} e^{-ik_\mu x^\mu} u(k^\mu)$ for which $\partial_\mu \psi = -ik_\mu \psi$

The Dirac equation then becomes: $(i\hbar\gamma^\mu k_\mu - mc)\psi = 0$ which is algebraic!

After some work one can show that $k^\mu = \pm\frac{1}{\hbar}p^\mu$ and we are left with 4 solutions:

$$\psi^{(1)} = Ae^{i\frac{p_\mu x^\mu}{\hbar}}\begin{pmatrix} E/mc^2 - p_z/mc \\ -p_x/mc - ip_y/mc \\ 1 \\ 0 \end{pmatrix} \qquad \psi^{(2)} = Ae^{i\frac{p_\mu x^\mu}{\hbar}}\begin{pmatrix} -p_x/mc + ip_y/mc \\ E/mc^2 + p_z/mc \\ 0 \\ 1 \end{pmatrix}$$

$$\psi^{(3)} = Ae^{-i\frac{p_\mu x^\mu}{\hbar}}\begin{pmatrix} 0 \\ 1 \\ -p_x/mc + ip_y/mc \\ -E/mc^2 + p_z/mc \end{pmatrix} \qquad \psi^{(4)} = Ae^{-i\frac{p_\mu x^\mu}{\hbar}}\begin{pmatrix} 1 \\ 0 \\ -E/mc^2 - p_z/mc \\ -p_x/mc - ip_y/mc \end{pmatrix}$$

Note:
Typically we write:

$$\psi^{(1)} = Ae^{i\frac{p_\mu x^\mu}{\hbar}}u^{(1)}$$
$$\psi^{(2)} = Ae^{i\frac{p_\mu x^\mu}{\hbar}}u^{(2)}$$
$$\psi^{(3)} = Ae^{-i\frac{p_\mu x^\mu}{\hbar}}v^{(1)}$$
$$\psi^{(4)} = Ae^{-i\frac{p_\mu x^\mu}{\hbar}}v^{(2)}$$

where $u^{(1)}, u^{(2)}$ are particle spinors and $v^{(1)}, v^{(2)}$ are anti-particle spinors

Note: $\psi^{(i)} \to \psi^{(i)}_{rest}$ w/ $\vec{p} = 0$ (you fill in the details in the HW)

These are quite different than the "decoupled" spinors in NR QM, i.e. $\psi(x)\chi$ w/ $\chi = \alpha\binom{0}{1} + \beta\binom{1}{0}$.
Here the energy and momentum dependence cannot be extracted as an overall coefficient like $\psi(x)$.

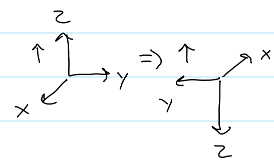
Looking again at S_z: $S_z\psi^{(1)} = \frac{\hbar}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} Ae^{i\frac{p_\mu x^\mu}{\hbar}}\begin{pmatrix} E/mc^2 - p_z/mc \\ -p_x/mc - ip_y/mc \\ 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}Ae^{i\frac{p_\mu x^\mu}{\hbar}}\begin{pmatrix} E/mc^2 - p_z/mc \\ p_x/mc + ip_y/mc \\ 1 \\ 0 \end{pmatrix} \not\propto \psi^{(1)}$ So not an eigenstate of S_z. Neither are other $\psi^{(i)}$

Unless we choose $\vec{p} = p_z\hat{k}$, then: $S_z\psi^{(1)} = \frac{\hbar}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} Ae^{i\frac{p_\mu x^\mu}{\hbar}}\begin{pmatrix} E/mc^2 - p_z/mc \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\psi^{(1)}$

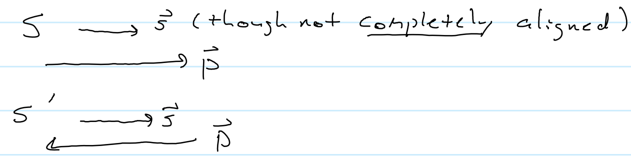
So it is often useful to work in terms of eigenstates of spin along the direction of motion, i.e. $S_{\vec{p}}$.
These are referred to as helicity states. We could have also set $\vec{p} = p_x\hat{i}$ and used S_x, similarly for y.

Characterizing particle states w/ helicity is almost just like characterizing them by S_z.

For instance if a particle has $S_z = +\frac{\hbar}{2}$, we can always rotate our coordinates so that the same particle has $S_z = -\frac{\hbar}{2}$. It is still a useful classification if we stick to one coordinate system.



Helicity is similar. If we have $S_{\vec{p}} = +\frac{\hbar}{2}$ then



But we can always boost to frame reversing $\vec{p}$ then giving us $S_{\vec{p}} = -\frac{\hbar}{2}$.

Except when the particle in question is massless! In that case there is no way to reverse $\vec{p}$ with a boost. So for massless particles, their helicity is an unchangeable intrinsic property (just like their total spin).

In fact for a given massless particle type (flavor) we might as well think of the $S_{\vec{p}} = \pm\frac{\hbar}{2}$ states as different particles!

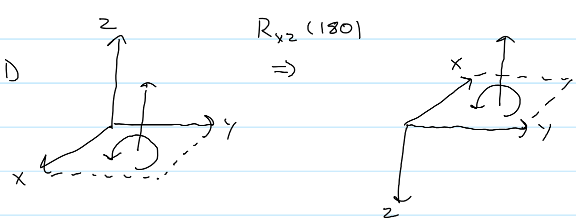
This has many implications, but first let's go back to our counting of states à la Wigner.

Recall we classify intrinsic spin states by the transformations that leave P^μ invariant.
For $m > 0$, we can work w/ $P^\mu = (mc, 0, 0, 0) \Rightarrow$ 3D rotations $\Rightarrow$ spin-$\frac{1}{2} \Rightarrow 2$ states.
However for $m = 0$ there is no rest frame. There is a simple P^μ to work with (remember the counting is independent of P^μ so we can choose any one that is handy).

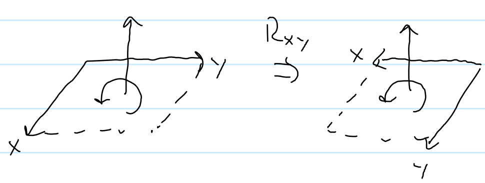
Consider: $P^\mu = \left(\frac{E}{c}, \frac{E}{c}, \underbrace{0, 0}\right)$ Note: $P_\mu P^\mu = 0$ as expected for $m = 0$.

This is only invariant under 2D rotations! But these cannot change the spin in this plane!

Example: 3D



But in 2D



Is any of this reflected in the Dirac equation?

Recall that w/ our conventions a boost on spinors is generated by $\sigma^{0i} = \frac{i}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}$

while a rotation on spinors is generated by $\sigma^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$

So if we take our 4-component $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$ we have that $\psi_\pm$ transform oppositely under boosts and alike under rotations.

The 2-component $\psi_\pm$ are called Weyl or <u>chiral</u> spinors.

The Dirac Lagrangian can be written: $\mathcal{L}_{\text{Dirac}} = (\hbar c)\bar{\psi}\gamma^\mu\partial_\mu\psi + mc^2\bar{\psi}\psi$

$$= -\hbar c\left(i\psi_-^\dagger\partial_\mu\sigma^\mu\psi_- + i\psi_+^\dagger\partial_\mu\bar{\sigma}^\mu\psi_+\right) + mc^2\left(\psi_-^\dagger\psi_+ + \psi_+^\dagger\psi_-\right)$$

$\sigma^\mu = (I, \sigma^i)$ $\qquad \bar{\sigma}^\mu = (I, -\sigma^i)$ (You get to do in HW)

Recall: $\bar{\psi} = i\psi^\dagger\gamma^0$ $\gamma^0 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ So for $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \Rightarrow \bar{\psi} = (\psi_-^\dagger \;\; \psi_+^\dagger)$

Now the important thing to note is that if $m \neq 0$ we need <u>both</u> ψ_+ and ψ_- (hence a 4-comp. Dirac spinor). However if $m=0$, we can actually work with just one of ψ_+ or ψ_-, i.e. 2 component Weyl spinors which satisfy:

$$\left.\begin{array}{l} i\bar{\sigma}^\mu\partial_\mu\psi_+ = 0 \\ \underline{\underline{\text{or}}} \quad i\sigma^\mu\partial_\mu\psi_- = 0 \end{array}\right\}$$ Weyl equations. Each one describes a particle/anti-particle pair, hence 2 real d.o.f.

Choosing to work with ψ_+ or ψ_- for <u>massless</u> spinors is exactly the same as working with positive or negative helicity states! To see this consider one example:

$$\psi^{(1)} = Ae^{\frac{ip_\mu x^\mu}{\hbar}}\begin{pmatrix} \frac{E}{mc^2} - \frac{p_z}{mc} \\ -\frac{p_x}{mc} - \frac{ip_y}{mc} \\ 1 \\ 0 \end{pmatrix} \xrightarrow[\vec{p} = p\hat{k}]{} \underbrace{Ae^{\frac{ip_\mu x^\mu}{\hbar}}\begin{pmatrix} \frac{E}{mc^2} - \frac{p}{mc} \\ 0 \\ 1 \\ 0 \end{pmatrix}} \xrightarrow[m=0]{} \underbrace{Ae^{\frac{ip_\mu x^\mu}{\hbar}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}$$

$m=0 \Downarrow E^2 - p^2c^2 = 0 \Downarrow E = pc$

Eigenstate of $S_{\vec{p}}$ since $S_{\vec{p}} = S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$

But not a state of pure chirality, e.g. $\begin{pmatrix} \psi_+ \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ \psi_- \end{pmatrix}$

Eigenstate of $S_{\vec{p}}$ and pure chirality $\begin{pmatrix} 0 \\ \psi_- \end{pmatrix}$

Everything we have done so far has been illustrated with our conventions for the γ's, but it may not be obvious that it works for other choices, e.g. when the γ's do not split like $-i\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$.

This is where γ^5 enters. For our conventions $\gamma^5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (Recall $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$)

And we can use it to form projection operators $P_\pm = \frac{1}{2}(1 \pm \gamma^5) \Rightarrow P_+\psi = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \psi_+$

$$P_-\psi = \psi_-$$

(HW)

But from the definition of γ^5, we can show that $\frac{1}{2}(1 \pm \gamma^5)$ is a projection operator in any representation of γ's.

So instead of using the (representation dependent) split $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$ we can just <u>define</u> $\psi_+ = P_+\psi$, $\psi_- = P_-\psi$

Here again we can see the difference between chirality and helicity:

$P_\pm = \frac{1}{2}(1 \pm \gamma^5)$ projects onto states of definite chirality.

Parity is a discrete transformation that takes $P: x^0 \to x^0, \underbrace{x^i \to -x^i}$. It is <u>not</u> a part of $SO(1,3)$!

alternatively you could just reflect a single axis, but <u>not</u> two.

An important aspect of parity is that it leaves the sign of rotations R_{yz}, R_{zx}, R_{xy} invariant, but reverses the sign of boosts B_{tx}, B_{ty}, B_{tz}.

Looking back at how we defined $\psi_\pm$, i.e. transforming w/ σ^k for rotations but $\pm\sigma^i$ for boosts, we can immediately infer that parity interchanges these, i.e. $P\psi_\pm = \psi_\mp$

But remembering the form of $\gamma^0 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we can then say $P = i\gamma^0$.

For the massless case, this is what we expect for helicity states,

$$\begin{array}{ccccc} \longrightarrow \vec{s} & & & \longrightarrow \vec{s} & \\ \longrightarrow \vec{p} & S_{\vec{p}} = +\frac{\hbar}{2} & \overset{P}{\Longrightarrow} & \vec{p} \longleftarrow & S_{\vec{p}} = -\frac{\hbar}{2} \\ \psi_+ & & & \psi_- & \end{array}$$

Finally we can take parity and build two more types of quantities in addition to scalars, vectors, etc.

We know: $\bar{\psi}\psi$ - scalar under $P: \bar{\psi}\psi \to \bar{\psi}(-i\gamma^{0\dagger})(i\gamma^0)\psi = \bar{\psi}(i\gamma^0)(i\gamma^0)\psi = \bar{\psi}\psi$

$\bar{\psi}\gamma^\mu\psi$ - vector $\quad \bar{\psi}\gamma^\mu\psi \to \left\{\begin{array}{ll} \bar{\psi}\gamma^0\psi & \mu=0 \\ -\bar{\psi}\gamma^i\psi & i=i \end{array}\right\}$ as expected for a vector

Then we can form: $\bar{\psi}\gamma^5\psi$ which transforms under $P: \bar{\psi}\gamma^5\psi \to \bar{\psi}(i\gamma^0)\gamma^5 i\gamma^0\psi = -\bar{\psi}\gamma^5\psi$ pseudoscalar

$\bar{\psi}\gamma^5\gamma^\mu\psi \qquad P: \bar{\psi}\gamma^5\gamma^\mu\psi \to \left\{\begin{array}{l} -\bar{\psi}\gamma^5\gamma^0\psi \\ \bar{\psi}\gamma^5\gamma^i\psi \end{array}\right\}$ pseudo (axial) - vector

In fact $\bar{\psi}\psi, \bar{\psi}\gamma^5\psi, \bar{\psi}\gamma^\mu\psi, \bar{\psi}\gamma^5\gamma^\mu\psi, \bar{\psi}\sigma^{\mu\nu}\psi$

1 1 4 4 6 = 16 independent matrices in spin space!

Solving Dirac

$$\gamma^\mu \partial_\mu \psi + \frac{mc}{\hbar}\psi = 0 \qquad \text{Assume: } \psi(x^\mu) = A e^{i k_\mu x^\mu} u(k) \Rightarrow \partial_\mu \psi = i k_\mu \psi$$

$$\gamma^\mu \partial_\mu \psi + \frac{mc}{\hbar}\psi = i\gamma^\mu k_\mu \psi + \frac{mc}{\hbar}\psi = 0 \quad \text{or} \quad (i\hbar\gamma^\mu k_\mu + mc)\psi = 0$$

$$\gamma^\mu k_\mu = \gamma^0 k_0 + \gamma^i k_i = -i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}k_0 + (-i)\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}k_i = -i\begin{pmatrix}0 & k_\mu\sigma^\mu\\ k_\mu\bar\sigma^\mu & 0\end{pmatrix} \qquad \sigma^\mu = I + \vec\sigma, \quad \bar\sigma^\mu = I - \vec\sigma$$

$$(i\hbar\gamma^\mu k_\mu + mc)\begin{pmatrix}u_A\\ u_B\end{pmatrix} = \begin{pmatrix}mc\,u_A + \hbar k_\mu\sigma^\mu u_B\\ \hbar k_\mu\bar\sigma^\mu u_A + mc\,u_B\end{pmatrix} = 0 \Rightarrow \begin{aligned} u_A &= -\frac{\hbar}{mc}k_\mu\sigma^\mu u_B = \left(\frac{\hbar}{mc}\right)^2 k_\mu\sigma^\mu k_\nu\bar\sigma^\nu u_A\\ u_B &= -\frac{\hbar}{mc}k_\mu\bar\sigma^\mu u_A = \left(\frac{\hbar}{mc}\right)^2 \underbrace{k_\mu\sigma^\mu k_\nu\bar\sigma^\nu u_B} \end{aligned}$$

Plugging into each other.

$$\left.\begin{aligned} k_\mu\sigma^\mu &= k_0 I + \vec k\cdot\vec\sigma\\ k_\mu\bar\sigma^\mu &= k_0 I - \vec k\cdot\vec\sigma \end{aligned}\right\} k_\mu\sigma^\mu k_\nu\bar\sigma^\nu = k_0^2 I^2 - (\vec k\cdot\vec\sigma)^2$$

But: $\vec k\cdot\vec\sigma = \begin{pmatrix}0 & k_x\\ k_x & 0\end{pmatrix} + \begin{pmatrix}0 & -ik_y\\ ik & 0\end{pmatrix} + \begin{pmatrix}k_z & 0\\ 0 & -k_z\end{pmatrix} = \begin{pmatrix}k_z & k_x - ik_y\\ k_x + ik_y & -k_z\end{pmatrix}$

So: $(\vec k\cdot\vec\sigma)^2 = \begin{pmatrix}k_z & k_x - ik_y\\ k_x + ik_y & -k_z\end{pmatrix}\begin{pmatrix}k_z & k_x - ik_y\\ k_x + ik_y & -k_z\end{pmatrix} = \begin{pmatrix}k_z^2 + k_x^2 + k_y^2 & 0\\ 0 & k_z^2 + k_x^2 + k_y^2\end{pmatrix} = k^2 I$

Then: $u_A = \left(\frac{\hbar}{mc}\right)^2 (k_0^2 I^2 - k^2 I^2) u_A \Rightarrow -m^2c^2 = \hbar^2(k_0^2 - k^2) \Rightarrow \hbar k_\mu = \pm p_\mu$ since $p_\mu p^\mu = -p_0^2 + \vec p^{\,2} = -m^2c^2$

Going back:

$$u_A = -\frac{1}{mc}(\pm p_\mu\sigma^\mu)u_B = \left(\mp\frac{p_0}{mc}I \mp \frac{1}{mc}\vec\sigma\cdot\vec p\right)u_B = \begin{pmatrix}\pm\frac{E}{mc^2} \mp \frac{p_z}{mc} & \mp\left(\frac{p_x}{mc} - i\frac{p_y}{mc}\right)\\ \mp\left(\frac{p_x}{mc} + i\frac{p_y}{mc}\right) & \pm\frac{E}{mc^2} \pm \frac{p_z}{mc}\end{pmatrix}u_B$$

$$u_B = -\frac{1}{mc}(\pm p_\mu\bar\sigma^\mu)u_A = \left(\mp\frac{p_0}{mc}I \pm \frac{1}{mc}\vec\sigma\cdot\vec p\right)u_A = \begin{pmatrix}\pm\frac{E}{mc^2} \pm \frac{p_z}{mc} & \pm\left(\frac{p_x}{mc} - i\frac{p_y}{mc}\right)\\ \pm\left(\frac{p_x}{mc} + i\frac{p_y}{mc}\right) & \pm\frac{E}{mc^2} \mp \frac{p_z}{mc}\end{pmatrix}u_A$$

Choosing: $u_B = \begin{pmatrix}1\\ 0\end{pmatrix}$ w/ lower sign $\Rightarrow u_A = \begin{pmatrix}-\frac{E}{mc^2} + \frac{p_z}{mc}\\ \frac{p_x}{mc} + i\frac{p_y}{mc}\end{pmatrix} \Rightarrow \psi^{(i)} = Ae^{-i\frac{p_\mu x^\mu}{\hbar}}\begin{pmatrix}-\frac{E}{mc^2} + \frac{p_z}{mc}\\ \frac{p_x}{mc} + i\frac{p_y}{mc}\\ 1\\ 0\end{pmatrix}$ $\quad \psi^{(i)}_{rest} = Ae^{i\frac{mc^2}{\hbar}t}\begin{pmatrix}-1\\ 0\\ 1\\ 0\end{pmatrix}$

$u_B = \begin{pmatrix}0\\ -1\end{pmatrix}$ w/ lower sign $\Rightarrow u_A = \begin{pmatrix}-\frac{p_x}{mc} + i\frac{p_y}{mc}\\ \frac{E}{mc^2} + \frac{p_z}{mc}\end{pmatrix} \Rightarrow \psi^{(ii)} = Ae^{-i\frac{p_\mu x^\mu}{\hbar}}\begin{pmatrix}-\frac{p_x}{mc} + i\frac{p_y}{mc}\\ \frac{E}{mc^2} + \frac{p_z}{mc}\\ 0\\ -1\end{pmatrix}$ $\quad \psi^{(ii)}_{rest} = Ae^{i\frac{mc^2}{\hbar}t}\begin{pmatrix}0\\ 1\\ 0\\ -1\end{pmatrix}$

$u_A = \begin{pmatrix}1\\ 0\end{pmatrix}$ w/ upper sign $\Rightarrow u_B = \begin{pmatrix}\frac{E}{mc^2} + \frac{p_z}{mc}\\ \frac{p_x}{mc} + i\frac{p_y}{mc}\end{pmatrix} \Rightarrow \psi^{(iii)} = Ae^{i\frac{p_\mu x^\mu}{\hbar}}\begin{pmatrix}1\\ 0\\ \frac{E}{mc^2} + \frac{p_z}{mc}\\ \frac{p_x}{mc} + i\frac{p_y}{mc}\end{pmatrix}$ $\quad \psi^{(iii)}_{rest} = Ae^{-i\frac{mc^2}{\hbar}t}\begin{pmatrix}1\\ 0\\ 1\\ 0\end{pmatrix}$

$u_A = \begin{pmatrix}0\\ 1\end{pmatrix}$ w/ upper sign $\Rightarrow u_B = \begin{pmatrix}\frac{p_x}{mc} - i\frac{p_y}{mc}\\ \frac{E}{mc^2} - \frac{p_z}{mc}\end{pmatrix} \Rightarrow \psi^{(iv)} = Ae^{i\frac{p_\mu x^\mu}{\hbar}}\begin{pmatrix}0\\ 1\\ \frac{p_x}{mc} - i\frac{p_y}{mc}\\ \frac{E}{mc^2} - \frac{p_z}{mc}\end{pmatrix}$ $\quad \psi^{(iv)}_{rest} = Ae^{-i\frac{mc^2}{\hbar}t}\begin{pmatrix}0\\ 1\\ 0\\ 1\end{pmatrix}$